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CHANGE OF VARIABLE FORMULA FOR "LEBESGUE MEASURES" ON $\mathbb{R}^{\mathbb{N}}$

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Abstract

We prove change of variable formula for wide class of "Lebesgue measures" on $\mathbb{R}^{\mathbb{N}}$ and extend a certain result obtained in [1].

Let $\mathbb{R}^n (n > 1)$ be an n-dimensional Euclidean space and let μ_n be an n-dimensional standard Lebesgue measure on $\mathbb{R}^n$. Further, let T be a linear μ_n-measurable transformation of $\mathbb{R}^n$.

It is obvious that $\mu_n T^{-1}$ is absolutely continuous with respect to μ_n and there exists a non-negative μ_n-measurable function Φ on $\mathbb{R}^n$ such that

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$$\mu_n(T^{-1}(X)) = \int_X \Phi(y) d\mu_n(y),$$

for every μ_n -measurable subset X of $\mathbb{R}^n$.

The function Φ plays the role of the Jacobian $J(T^{-1})$ of the transformation T^{-1} (or, rather the absolute value of the Jacobian) (see, e.g., [3]) in the theory of transformations of multiple integrals. It is clear that $J(T^{-1})$ coincides with a Radon-Nikodym derivative $\frac{d\mu_n T^{-1}}{d\mu_n}$, which is unique, a.e., with respect to μ_n.

It is clear that

$$\frac{d\mu_n T^{-1}}{d\mu_n}(x) = \lim_{k \to \infty} \frac{\mu_n(T^{-1}(U_k(x)))}{\mu_n(U_k(x))} (\mu_n - \text{a.e.}),$$

where $U_k(x)$ is a spherical neighborhood with the center in $x \in \mathbb{R}^n$ and radius $r_k > 0$, so that $\lim_{k \to \infty} r_k = 0$. The class of such spherical neighborhoods generates the so-called Vitali differentiability class of subsets which allows us to calculate the Jacobian $J(T^{-1})$ of the transformation T^{-1}.

If we consider a vector space of all real-valued sequences $\mathbb{R}^{\mathbb{N}}$ (equipped with Tychonoff topology), then we observe that for the infinite-dimensional Lebesgue measure [1] (or [2]) defined in $\mathbb{R}^{\mathbb{N}}$ there does not exist any Vitali system of differentiability, but in spite of non-existence of such a system the inner structure of this measure allows us to define a form of the Radon-Nikodym derivative defined by any linear transformation of $\mathbb{R}^{\mathbb{N}}$. In order to show it, let consider the following.

Example 1. Let $\mathcal{R}_1$ be the class of all infinite dimensional rectangles $R \in B(\mathbb{R}^{\mathbb{N}})$ of the form

$$R = \prod_{i=1}^{\infty} R_i,\ R_i = (a_i,\ b_i),\ -\infty < a_i \le b_i < +\infty,$$

such that

$$0 \le \prod_{i=1}^{\infty} (b_i - a_i) < \infty.$$

Let τ_1 be the set function on $\mathcal{R}_1$ defined by

$$r_1(R) = \prod_{i=1}^{\infty} (b_i - a_i).$$

Baker [1] proved that the functional λ_1 defined by

$$(\forall X)\left(X \in B(\mathbb{R}^{\mathbb{N}}) \to \lambda_1(X) = \inf\left\{ \sum_{j=1}^{\infty} \tau_1(R_j) : R_j \in \mathcal{R}_1 \ \&\ X \subseteq \cup_{j=1}^{\infty} R_j \right\} \right)$$

is a quasi-finite[1] translation-invariant Borel measure in $\mathbb{R}^{\mathbb{N}}$.

The following change of variable formula has been established in [1] (cf. p. 1029):

Let $T^n : \mathbb{R}^n \to \mathbb{R}^n,\ n > 1,$ be a linear transformation with Jacobian $\Delta \ne 0,$ and let $T^{\mathbb{N}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the map defined by

$$T^{\mathbb{N}}(x) = (T^n(x_1,\ ...,\ x_n),\ x_{n+1},\ x_{n+2},\ ...),\ x = (x_i)_{i \in N} \in \mathbb{R}^{\mathbb{N}}. \qquad (1)$$

Then for each $E \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}),$ we have

$$\lambda_1(T^{\mathbb{N}}(E)) = |\,\Delta\,| \lambda_1(E).$$

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[1] A Borel measure μ defined in $\mathbb{R}^{\mathbb{N}}$ is called quasi-finite if $0 < \mu(U) < +\infty$ for any compact set $U \subset \mathbb{R}^{\mathbb{N}}$.

The main goal of the present paper is to extend change of variable formula presented in Example 1 for wide class of “Lebesgue measures” on $\mathbb{R}^{\mathbb{N}}$.

Now, let (X_i, M_i, μ_i) be a sequence of measure spaces and for each i let ρ_i be a metric on the set X_i. Assume that the following conditions are satisfied.

(i) Each (X_i, ρ_i) is a locally compact metric space.

(ii) Each M_i contains the family, $B(X_i)$, of Borel subsets of X_i and μ_i is a regular Borel measure on X_i (cf. [6], Def. 2.15).

(iii) For all i, and for every $\delta > 0$ there exists a sequence (A_j) of Borel subsets of X_i such that $d_i(A_j) > \delta$ and $X_i = \bigcup_{j=1}^{\infty} A_j$, where $d_i(A_j)$ is the diameter of A_j in X_i.

Define $X = \prod_{i=1}^{\infty} X_i$, and let equip X with the product topology. Let denote by $\mathcal{R}$ the family of all rectangles $R \subseteq X$ of the form $R = \prod_{i=1}^{\infty} R_i$, $R_i \in \mathcal{B}(X_i)$, and

$$0 \le \prod_{i=1}^{\infty} \mu_i(R_i) := \lim_{n\to\infty} \prod_{i=1}^{n} \mu_i(R_i) < +\infty.$$

For $R \in \mathcal{R}$, we define

$$\tau(R) = \prod_{i=1}^{\infty} \mu_i(R_i).$$

Let τ^* be the set function on the powerset $\mathcal{P}(X)$ defined by

$$\tau^*(E) = \inf\left\{\sum \tau(R_j) : R_j \in \mathbb{R} \ \& \ E \subseteq \bigcup R_j\right\}$$

for $E \subseteq X$.

Also, let use the convention that $0 \cdot +\infty = +\infty \cdot 0 = 0$, $+\infty \cdot +\infty = +\infty$, and that the infimum taken over the empty set has the value $+\infty$.

Theorem 1([2], Theorem I, p. 2579)**.** *The set function* τ^* *is an outer measure on X. Let M be the σ-algebra of subsets of X which are measurable with respect to* τ^*, *and let* μ *be the measure on M obtained by restricting* τ^* *to M. Then* $B(X) \subseteq M$, *and for all* $R = \prod_{i=1}^{\infty} R_i \in \mathcal{R}$, *we have* $\mu(R) = \prod_{i=1}^{\infty} \mu_i(R_i)$. *If each space* X_i *contains disjoint subsets* A_i, B_i *such that* $\mu_i(A_i) = \mu_i(B_i) = 1$, *then the measure* μ *is not σ-finite. Finally, assume that each* (X_i, ρ_i) *is an* M_i*-measurable group. If each* μ_i *is left-invariant measure on* M_i, *then* μ *is a left-invariant measure on M. Similarly, if each* μ_i *is right-invariant measure on* M_i, *then* μ *is right-invariant measure on M.*

Let $\mathcal{R}_2$ be the class of all infinite dimensional rectangles $R \in \mathbb{B}(\mathbb{R}^{\mathbb{N}})$ of the form

$$R = \prod_{i=1}^{\infty} R_i,\ R_i \in B(\mathbb{R})$$

such that

$$0 \le \prod_{i=1}^{\infty} m(R_i) := \lim_{n\to\infty} \prod_{i=1}^{n} m(R_i) < \infty,$$

where m denotes a one-dimensional classical Borel measure on $\mathbb{R}$.

Let τ_2 be a set function on $\mathcal{R}_2$, defined by

$$\tau_2(R) = \prod_{i=1}^{\infty} m(R_i).$$

As a simple consequence of Theorem 1, Baker [2] had obtained that the functional λ_2 defined by

$$(\forall X)\left(X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \to \lambda_2(X) = \inf\left\{\sum_{j=1}^{\infty} \tau_2(R_j) : R_j \in \mathcal{R}_2 \ \& \ X \subseteq \cup_{j=1}^{\infty} R_j\right\}\right)$$

is a quasi-finite translation-invariant Borel measure on $\mathbb{R}^{\mathbb{N}}$.

Baker [2] noted that the measure λ_2 is the final version of a similar measure which we introduced in the paper [1].

Remark 1. Baker [2] (cf. Definition 1, p. 2579) posed the problem whether λ_1 and λ_2 coincides. In [4] has been established that λ_2 is absolutely continuous with respect to the measure λ_1, and λ_1 and λ_2 are not equivalent.

Below we consider a certain example of infinite-dimensional Lebesgue measure, which extends the construction introduced in [2]. In order to do it we need the following simple auxiliary proposition.

Lemma 1. *Let $(n_i)_{i \in N}$ be the sequence of natural numbers. Then the following formula is valid*

$$\prod_{i \in N} \mathcal{B}(\mathbb{R}^{n_i}) = \mathcal{B}(\mathbb{R}^{\mathbb{N}}).$$

Theorem 2. *Let $(n_i)_{i \in N}$ be the sequence of natural numbers and let $\mathcal{R}_3$ be the class of all infinite dimensional rectangles $R \in B(\mathbb{R}^{\mathbb{N}})$ of the form*

$$R = \prod_{i=1}^{\infty} R_i,\ R_i \in B(\mathbb{R}^{n_i})$$

such that

$$0 \le \prod_{i=1}^{\infty} m_{n_i}(R_i) := \lim_{n \to \infty} \prod_{i=1}^{\infty} m_{n_i}(R_i) < \infty,$$

where m_{n_i} *denotes an* n_i*-dimensional classical Borel measure on* $\mathbb{R}^{n_i}$.

Let τ_3 *be a set function on* $\mathcal{R}_3$, *defined by*

$$\tau_3(R) = \prod_{i=1}^{\infty} m_{n_i}(R_i).$$

Then the functional λ_3 *defined by*

$$(\forall X)\left(X \in B(\mathbb{R}^{\mathbb{N}}) \to \lambda_3(X) = \inf\left\{ \sum_{j=1}^{\infty} \tau_3(R_j) : R_j \in \mathcal{R}_3 \ \& \ X \subseteq \cup_{j=1}^{\infty} R_j \right\}\right)$$

*is a quasi-*invariant *translation-invariant Borel measure on* $\mathbb{R}^{\mathbb{N}}$ *such that* λ_3 *is absolutely continuous with respect to the measure* λ_2. *Moreover, if there exists* i_0 *such that* $n_i = 1$ *for* $i > i_0$, *then* $\lambda_3 = \lambda_2$.

The proof of Theorem 2 employs Theorem 1 and Lemma 1.

The following example gives an example of "Lebesgue measure" on $\mathbb{R}^{\mathbb{N}}$ which does not coincides with the "Lebesgue measure" λ_2.

Example 2. Let $n_i = 2$ for $i \in \mathbb{N}$. We set

$$Y_i = \{(x_{2i+1}, x_{2(i+1)}) : (x_{2i+1}, x_{2(i+1)}) \in R^{n_i},$$
$$0 \le x_{2i+1} \le 1, x_{2(i+1)} \le 2 - 2x_{2i+1}\}.$$

It is obvious that $\lambda_3\left(\prod_{k \in \mathbb{N}} Y_i\right) = 1$.

We have shown that $\lambda_2\left(\prod_{k \in \mathbb{N}} Y_i\right) = +\infty$. Firstly, let show that the set $\prod_{k \in \mathbb{N}} Y_i$ is not covered by the union of countable family of elements of $\mathcal{R}_2$. Indeed, let assume the contrary and let for any family $(R_j)_{j \in \mathbb{N}}$ of elements of $\mathcal{R}_2$ we have $\prod_{k \in \mathbb{N}} Y_i \subseteq \cup_{j \in \mathbb{N}} R_j$. Then there exists $j_0 \in \mathbb{N}$ such that

$$0 < \lambda_2\left(R_{j_0} \cap \prod_{i \in \mathbb{N}} Y_i\right) < +\infty.$$

Since $R_{j_0} = \prod_{i \in N} (X_{2i+1} \times X_{2(i+1)})$, where $X_{2i+1}, X_{2(i+1)} \in B(R)$ and $0 < \prod_{i \in \mathbb{N}} m(X_i) < \infty$, we conclude that

(1) $\lim_{i \to \infty} m_i(X_i) = 1$;

(2) $\lim_{i \to \infty} (m \times m)((X_{2i+1} \times X_{2(i+1)}) \cap Y_i) = 1$.

For sufficient small $\varepsilon > 0$ there exists i_0 such that $1 - \varepsilon < m(X_i) < 1 + \varepsilon$ whenever $i > i_0$. Hence, we get

$$(m \times m)((X_{2i+1} \times X_{2(i+1)}) \cap Y_i) \leq \frac{(3 - \varepsilon) \times (1 + \varepsilon)}{4}$$

for $i > i_0$.

If we choose $\varepsilon > 0$ that

$$0 < \frac{(3 - \varepsilon) \times (1 + \varepsilon)}{4} \leq \frac{7}{8},$$

then we will obtain

$$(m \times m)((X_{2i+1} \times X_{2(i+1)}) \cap Y_i) \leq \frac{7}{8},$$

which contradicts to the condition 2).

Following our convention, the infimum taken over the empty set has the value $+\infty$, which implies that that $\lambda_2\left(\prod_{i \in \mathbb{N}} Y_i\right) = +\infty$. The latter relation means that the measures λ_2 and λ_3 are different.

In connection with Theorem 2 and Example 2 we state the following problems.

Problem 1. Describe the class of all sequences of natural numbers $(n_i)_{i \in \mathbb{N}}$ for which $\lambda_3 = \lambda_2$.

Problem 2. Let K be a class of infinite-dimensional Lebesgue measures in $\mathbb{R}^{\mathbb{N}}$ defined by Theorem 2. Does there exists a translation-invariant Borel measure in $\mathbb{R}^{\mathbb{N}}$ such that for every $\mu \in K$ and for every $D \in B(\mathbb{R}^{\mathbb{N}})$ with $0 \leq \mu(D) < \infty$ the following equality $\lambda(D) = \mu(D)$ holds?

In context with Problem 2 we state the following

Conjecture 1. The measure constructed in [5] is the solution of Problem 2.

Theorem 3. *Let* $(n_i)_{i \in \mathbb{N}}$ *be the sequence of natural numbers, and let* $T^{n_i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i > 1$, *be a family of linear transformations with Jacobians* $\Delta_i \neq 0$ *and* $0 < \prod_{i=1}^{\infty} | \Delta_i | < \infty$. Let $T^N : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ *be the map defined by*

$$T^{\mathbb{N}}(x) = (T^{n_1}(x_1, \ldots, x_{n_1}), T^{n_2}(x_{n_1+1}, \ldots, x_{n_1+n_2}), \ldots),$$

where $x = (x_i)_{i \in N} \in \mathbb{R}^{\mathbb{N}}$. *Then for each* $E \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$, *we have*

$$\lambda_3(T^N(E)) = \left(\prod_{i=1}^{\infty} | \Delta_i | \right) \lambda_3(E).$$

Proof. Taking into account an equality

$$T^{\mathbb{N}}(\mathcal{R}_3) = \mathcal{R}_3,$$

Lemma 1 and Theorem 2 , for $E \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$, we have

$$\lambda_3(T^{\mathbb{N}}(E)) = \inf \left\{ \sum_{j=1}^{\infty} \tau_3(R_j) : R_j \in \mathcal{R}_3 \ \& \ T^{\mathbb{N}}(E) \subseteq \cup_{j=1}^{\infty} R_j \right\}$$

$$= \inf \left\{ \sum_{j=1}^{\infty} \tau_3(R_j) : R_j = \prod_{i=1}^{\infty} R_i^{(j)} \right.$$

$$\left.\in \mathcal{R}_3 \ \& \ R_i^{(j)} \in \mathcal{B}(\mathbb{R}^{n_i}) \ \& \ T^{\mathbb{N}}(E) \subseteq \cup_{j=1}^{\infty} R_j \right\}$$

$$= \inf\left\{ \sum_{j=1}^{\infty} \tau_3(R_j) : R_j = \prod_{i=1}^{\infty} R_i^{(j)} \in \mathcal{R}_3 \ \& \ R_i^{(j)} \right.$$

$$\left. \in \mathcal{B}(\mathbb{R}^{n_i}) \ \& \ E \subseteq (T^{\mathbb{N}})^{-1}\left(\cup_{j=1}^{\infty} R_j\right) \right\}$$

$$= \inf\left\{ \sum_{j=1}^{\infty} \tau_3(R_j) : R_j = \prod_{i=1}^{\infty} R_i^{(j)} \in \mathcal{R}_3 \ \& \ R_i^{(j)} \right.$$

$$\left. \in \mathcal{B}(\mathbb{R}^{n_i}) \ \& \ E \subseteq \cup_{j=1}^{\infty} (T^{\mathbb{N}})^{-1}(R_j) \right\}$$

$$= \inf\left\{ \sum_{j=1}^{\infty} \tau_3\left(\prod_{i=1}^{\infty} R_i^{(j)} \right) : R_j = \prod_{i=1}^{\infty} R_i^{(j)} \in \mathcal{R}_3 \ \& \ R_i^{(j)} \in \mathcal{B}(\mathbb{R}^{n_i}) \ \& \right.$$

$$\left. E \subseteq \cup_{j=1}^{\infty} \left(\prod_{i=1}^{\infty} (T^{n_i})^{-1} R_i^{(j)} \right) \right\}$$

$$= \left(\prod_{i=1}^{\infty} |\Delta_i| \right) \inf\left\{ \sum_{j=1}^{\infty} \tau_3\left(\prod_{i=1}^{\infty} (T^{n_i})^{-1} R_i^{(j)} \right) : R_j \right.$$

$$= \prod_{i=1}^{\infty} R_i^{(j)} \in \mathcal{R}_3 \ \& \ R_i^{(j)} \in \mathcal{B}(\mathbb{R}^{n_i}) \ \&$$

$$\left. E \subseteq \cup_{j=1}^{\infty} \prod_{i=1}^{\infty} (T^{n_i})^{-1}(R_i^{(j)}) \right\}$$

$$= \left(\prod_{i=1}^{\infty} |\Delta_i| \right) \inf\left\{ \sum_{j=1}^{\infty} \tau_3(S_j) : S_j = \prod_{i=1}^{\infty} S_i^{(j)} \in \mathcal{R}_3 \ \& \right.$$

$$S_i^{(j)} = (T^{n_i})^{-1}(R_i^{(j)}) \in \mathcal{B}(\mathbb{R}^{n_i}) \ \& \ E \subseteq \cup_{j=1}^{\infty} \prod_{i=1}^{\infty} (T^{n_i})^{-1}(R_i^{(j)}) \Bigg\}$$

$$= \left(\prod_{i=1}^{\infty} |\Delta_i| \right) \inf \left\{ \sum_{j=1}^{\infty} \tau_3(S_j) : S_j \in \mathcal{R}_3 \ \& \ E \subseteq \cup_{j=1}^{\infty} S_j \right\}$$

$$= \left(\prod_{i=1}^{\infty} |\Delta_i| \right) \lambda_3(E).$$

Remark 2. Theorem 3 extends change of variable formula for the infinite-dimensional Lebesgue measure considered in Example 1. Indeed, let $T^n : \mathbb{R}^n \to \mathbb{R}^n$, $n > 1$, be a linear transformation with Jacobian $\Delta \neq 0$. Let $n_1 = n$ and $n_i = 1$ for $i > 1$. Further, we set $T^{n_1} = T^n$ and $T^{n_k} = I$, where $I : \mathbb{R} \to \mathbb{R}$ is an identity transformation of $\mathbb{R}$ defined by $I(x) = x$ for $x \in \mathbb{R}$.

Let a map $T^{\mathbb{N}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be defined by

$$T^{\mathbb{N}}(x) = (T^n(x_1, \ldots, x_n), x_{n+1}, x_{n+2}, \ldots), \ x = (x_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

Then, by Theorems 1–2, for $T^{\mathbb{N}}$ and for each $E \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$, we have

$$\lambda_2(T^{\mathbb{N}}(E)) = \lambda_3(T^{\mathbb{N}}(E)) = |\Delta| \lambda_3(E) = |\Delta| \lambda_2(E).$$

Remark 3. For $1 \leq i \leq 3$, we denote by S_i a class of all Borel subsets in R^N, which can be covered by the union of countable family of elements of $\mathbb{R}_i$. Then we have

(1) $S_1 \subset S_2 \subset S_3$;

(2) $\lambda_3(X) = \lambda_2(X) = \lambda_1(X)$ for $X \in S_1$;

(3) $\lambda_3(Y) = \lambda_2(Y)$ for $Y \in S_2$;

(4) $\lambda_3 \ll \lambda_2 \ll \lambda_1$, where the symbol "$\ll$" as usual, denotes an absolutely continuity of measures.

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